SOLID PERSPECTIVE AS A TOOL FOR THE GENERATION AND THE STUDY OF QUADRIC SURFACES

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ABSTRACT

It is common knowledge how the projection from a central point onto a plane can be used to generate conic sections as transformations of the circle. And how these transformations can be carried out, in graphic form, with a simple and repetitive procedure. The creation of the conics as plane sections of the cone requires a more advanced level of knowledge and graphic ability, since it involves the use of descriptive geometry. With the advent of the information technology and the ensuing possibility of constructing virtual spaces of three dimensions, the technology offers today to the researcher and the teacher the possibility to extend the above said constructions to the space. In this paper, we first describe the solid perspective in its theoretical basis and in its workability. In particular, we determine the bi-univocal perspective relationship between two spaces: the real space, isotropic, and the contracted and anisotropic space of the solid perspective. Francesco Borromini's Palazzo Spada Gallery is taken as case study to highlight how this perspective machine is capable of transforming architecture of regular shapes into the three-dimensional scenography of the same, and vice versa. We then present a sphere, studying its projective transformations into ellipsoid, paraboloid and hyperboloid. These transformations are finally examined from the canonical point of view of the projective geometry. Nowadays, thanks to the digital representation, it is possible to experiment directly in space the projective genesis of the ruled quadrics. Given two sheaves of planes corresponding in a perspectivity in space, these determine a surface which is a ruled hyperboloid or a quadric cone. It is possible to untie the two sheaves and freely move them about in space observing that the projectivity is preserved and that the two projective sheaves, in their new positions, determine a new ruled quadric.

THE PAPER ITSELF

SOLID PERSPECTIVE

The first theme concerns a generalization of the perspective by means of the solid homology. The digital modellers have, as it is known, some object deformation tools. These tools, called deformers, perform twisting, tapering and other parametric transformations. It is also possible to program the transformation that you want to perform and, consequently, it is possible to program the genesis of a plane perspective and a solid perspective.

As it is well known, the solid perspective depends on the position of the observer, on the position of the plane of collineation (or plane of the trace) and of the position of the limit plane (or vanishing plane). The perspective, three-dimensional, is in this case projected into the semi-space delimited by the vanishing plane, which contains the observer. The entities, which are at an indefinite distance ahead of the observer, are projected onto the limit plane, whereas the entities that are on the plane of collineation are projected in themselves, producing perspectives coincident with the same entity. Let's now imagine to fix the position of the observer and the position of the plane of trace in respect of any architecture or part of it, like a room.

We can imagine to move the limit plane from the distant boundary of the space toward the observer, following the main direction and to assist to the forming of a solid perspective, which will imperceptibly defer from the real object during the first steps of the displacement and which will warp more and more until it assumes, finally, the perspective view defined by the position assumed by the limit plane. During this transformation the observer will not be able to perceive any change of the linear perspective of the real-space observed, whereas he can see a slow colour change of the surface, owing to the variation of the angle of incidence of the sunrays. But we can also imagine, at this point, a translation of the plane of collineation, until it coincides with the limit plane. In this transformation, the solid perspective will be squeezed until the third dimension is effaced and until it crushes onto the picture plane.

These two passages show, therefore, the continuous transformation of the real space into a three-dimensional perspective space and the transformation of this last into a plane perspective.

The relationships that tie the coordinates of any point of the space to the coordinates of the corresponding point of the solid perspective can so be established. Let's begin by establishing what relationships exist between an object in real space and its solid perspective. Given a centre of projection **O'** in real space ∑ and a space ∑**'**, the support of the representation, which we will call scenographic space. This space is endless and superimposed on real space, but, for now, let's consider only a small part, the part between the two planes **π'** and **t**, both vertical and placed in front of the observer (fig. 1).

The first plane, also called the vanishing plane (plane of the vanishing points and

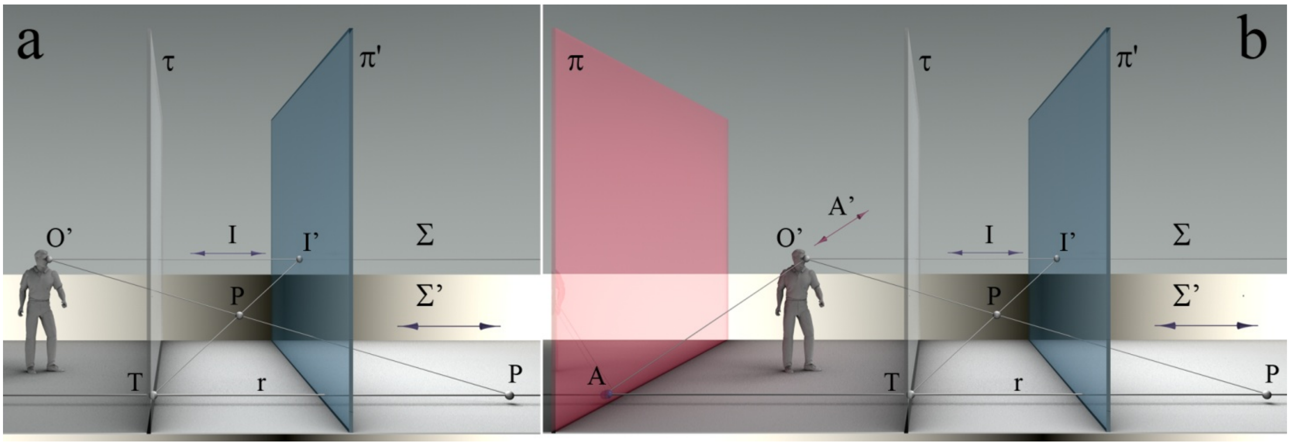

Fig. 1 – Projection of geometric figures belonging from a Σ space from center O' on a Σ' space creates a biunivocal correspondence between the two spaces. S is a isotropic space, S' is a anisotropic space, scenographic space.

vanishing lines) or first limit plane, will host the projections of the points of real space which are at an immeasurable distance, namely the projections of the so-called "points at infinity": The other plane, also called the plane of the traces (plane of the traces of the straight lines and of the traces of the planes) or plane of collineation, will host the points shared by real and scenographic space. Any straight line **r** of real space, so long as it is not parallel to the said planes, will then have as image the straight line **r'** determined by the points **T** and **I'**, where **T** is the point in which **r** meets the plane of the traces, while **I'** is the vanishing point, in other words the image of the direction of **r**, projected from **O'** on **π'** by means of a parallel to **r**. The segment **TI'** of **r'** therefore represents the portion of the straight line **r** that runs from point **T** to the direction **I** (or, if you prefer a less accurate but more expressive convention, that runs from point **T** to the "point at infinity" **I**).

Well , as we said earlier, the two spaces ∑ and ∑' are both prolonged ad infinitum and therefore any point **P** of the straight line **r**, will have a projection **P'** on the straight line **r'** and will be the point in which the projecting straight line **O'P** will meet **r'**. The straight line **r** is therefore fully represented by the straight line **r'**, entirely, that is, also in the parts that lie outside the portion of space included between the plane of the traces and the vanishing plane. Vice versa, to every point **Q'** of the straight line **r'** corresponds a point **Q** of the straight line **r**, the one detached, on **r**, by the projecting straight line **O'Q'**. In short, this is expressed by saying that the correspondence between **r** and **r'** is bi-univocal. And this correspondence is the same for all the points of the two straight lines, both prolonged ad infinitum in space.

Fig. 2 – A building immerse in the Σ space is projected in the Σ' space in the solid perspective visible at the picture. It is a scenographic design similar to the more famous one by Bramante, Borromini and others artist.

If we now consider a certain number of equal intervals, detached on **r**, we can observe how on **r'** the same number of intervals in perspective progression correspond to these intervals. This describes, albeit briefly, the different characteristics of the two spaces ∑ and ∑**'**, indefinitely extended and superimposed, but the first objective space is isotropic, while the second scenographic space is anisotropic.

Let's decide to call the focal distance **f**, namely the distance of **O'** from the vanishing plane **π'** (or first limit plane) and **d** the distance between the plane of the traces τ (or of collineation) and the vanishing plane **π'**, which is an important characteristic of the solid perspective because it defines the anisotropy, or better the compression of the scenographic space ∑**'** in the interval between the two planes and the latter's dilation outside the planes. If we take as a reference an **XYZ** system originating in **O'** and the axis **Z** perpendicular to the planes τ and π', the relationships between the **x'**, **y'**, **z'** coordinates of a point of the scenographic space and the coordinates **x**, **y**, **z**, of a point in real space, are as follows [1]:

$x' = (f \times x)/(d + z)$;

$y' = (f \times y)/(d + z)$;

$z' = (f \times z)/(d + z)$.

These relationships make it possible to rapidly create the solid perspective of even a complex object. Let's consider, for example, the model of Galleria Spada (fig. 2b). Numerical (or polygonal) representation programs often have tools, called "deformers" or "modifiers" that allow formulas to be used to transform the object. For example, by inserting the aforesaid expressions in a 4D Cinema

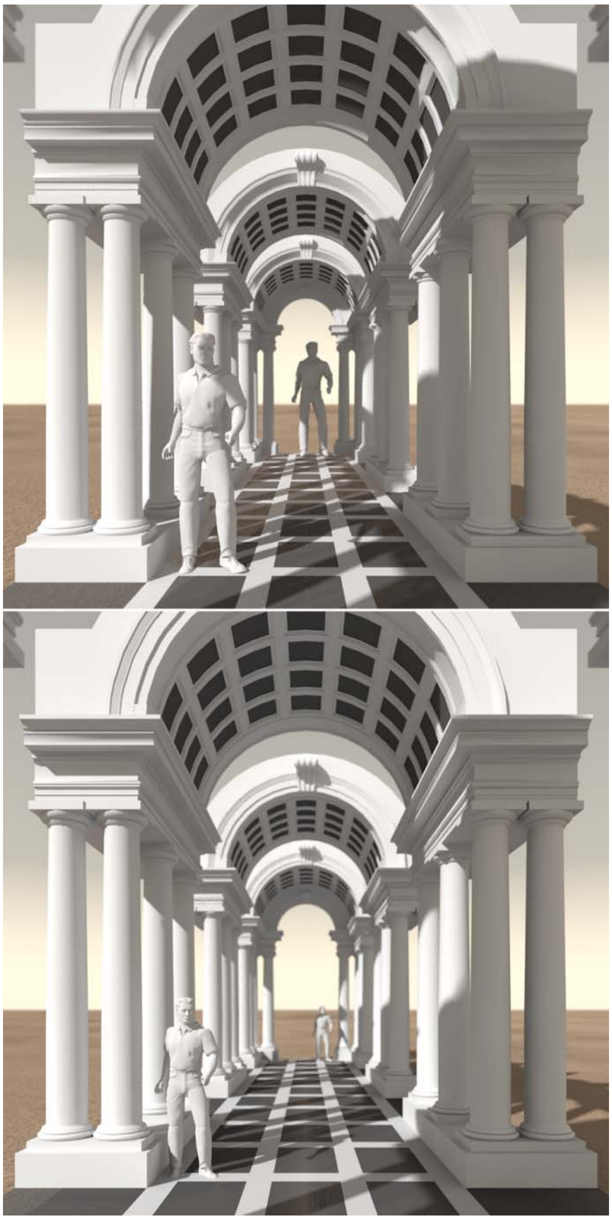

Fig. 3 - Two perspectives of the models of the Galleria Spada: the isotropic space Σ and the anisotropic Σ'

Formula (Objects / Deformation), you immediately get the scenographic design of the barn (fig. 2a). The model can also be exported for further elaboration. And the most immediate of these elaborations is the verification of the convergence of the edges that were parallel in the real world, in a vanishing point, in the illusory world of solid perspective (fig. 3). Let's suppose we translate the first limit plane, until it coincides with the plane of collineation: the result is a plane perspective. To carry out this deformation, all you have to do is to cancel d and equate:

f' =f – d

where f is the original focal length and f' is the new focal length, modified after translating the vanishing plane on the plane of the traces. So to use words familiar to those who study perspective, the vanishing plane and the plane of the traces now merge in what is commonly called the picture plane.

Using this procedure we obtained a traditional perspective by eliminating depth, similar to what happens when we create computer generated images, be they central or parallel. To fully understand these possibilities, we have to introduce into the perspective machine that we have created, a second limit plane **π**, placed behind the back of the observer at the distance we have called **d** and that also separates the plane of collineation from the first limit plane **π'**. In this manner, the perspective machine assumes all the values of a solid homology, or better, of the relationship, more in general, that exist between two superimposed spaces. Let's see how (fig.1).

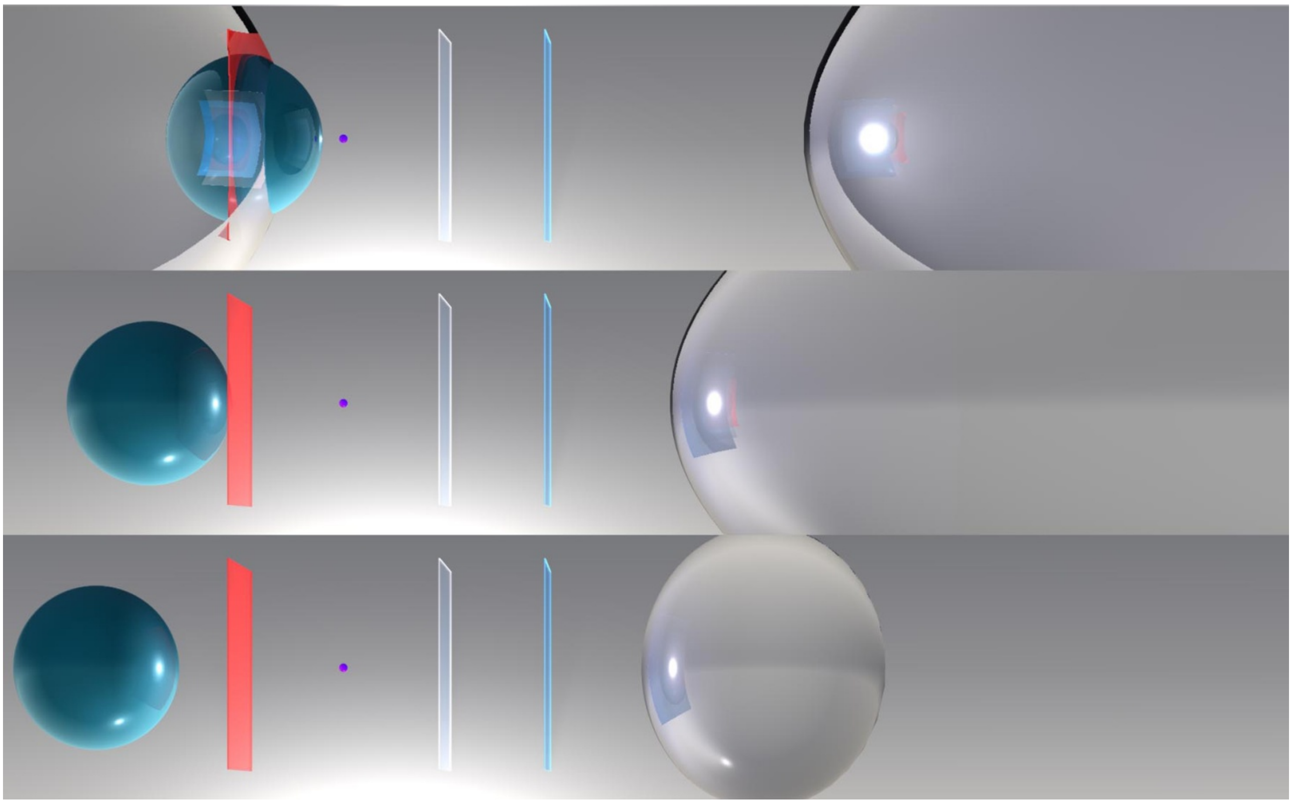

Fig. 4 - Illustrations of the metamorphoses of the solid projection of a sphere that draws closer toh e second limite plane, touches it and crosses it. The metamorphoses of the sphere are transparent, making it possibile to see the homological relationship.

Earlier we established the bi-univocal nature of the correspondence between the straight line **r** and **r'**, demonstrating how to any point **P** of **r** corresponds a point **P'** of **r'** and vice versa. Let's now consider point **A** that the straight line **r** have in common with the second limit plane, **π**. It's easy to demonstrate that the straight line **O'A** is parallel to **r'** and that, therefore, to point A of the straight line **r** corresponds the direction **A'** of the straight line **r'**. To make this correspondence more expressive and understandable, we can compare the two limit planes to the gates of a spaceport used by spaceships that can reach the outermost ends of the Universe. The first limit plane is the gate used by passengers arriving from outer space; the second limit plane is the gate used by passengers departing for outer space. There is no better way to experiment the properties of these gates than to pass through them: unfortunately however, we can only pass through **π**, the second one, the one that belongs to our ∑ space, since the first is immersed in an illusory space, i.e., scenographic space ∑**'**.

Let's consider, for example, a sphere **Ω**, and let's follow it or its image or transformation **Ω'** as it is gets closer to the second limit plane (fig. 4). As **Ω** gets closer to **π**, **Ω'** starts to look like (and to take on its geometric properties) an increasingly elongated ellipsoid. When **Ω** touches the plane **π**, in the contact point **V**, the corresponding point **V'** of **Ω'** leaves for deep space and the ellipsoid changes into a paraboloid. Now **Ω** is crossing **π** and an entire class of its points is projected at an incommensurable distance, therefore the paraboloid breaks into the two sheets of a hyperboloid. Finally, the **Ω** crosses the second limit plane, but before leaving it, it still touches it in one point, opposite to the first and **Ω'**, consequently, changes again into a paraboloid before becoming an ellipsoid again. At this point I would like to warn those who would like to test these transformations of one small problem that can be explained by the radical difference between our abstract reasoning and banal mechanical calculations. Actually, if you want your computer to represent the projections of a figure that crosses the second limit plane, all it can do is to signal an insurmountable error: machines cannot comprehend infinity. They only deal with finite quantities, either large or small. Infinity is a metaphysical concept. We solved this problem by creating a thin volume (not visible in the figure) that coincides with plane **π**, and I subtracted this volume from the objects that cross the limit. So, in the example of the sphere, in its calculations the machine can ignore the points that, in theory, are travelling in deep space.

PROJECTIVE GENESIS OF THE RULED QUADRICS

These transformations are finally examined from the canonical point of view of the projective geometry. Nowadays, thanks to the digital representation, it is possible to experiment directly in space the projective genesis of the ruled quadrics. Given two sheaves of planes corresponding in a perspectivity in space, these determine a surface which is a ruled hyperboloid or a quadric cone. It is possible to untie the two sheaves and freely move them about in space observing that the projectivity is preserved and that the two projective sheaves, in their new positions, determine a new ruled quadric. The ruled surfaces of second order (quadric) can also be generated by projective, through the correspondence of two forms of the first species linked by a projectivity. In fact [2]:

- two sheaf of planes in projective correspondence are cut in straight lines of a ruled surface;
- two pencil (of line segment range) in projective correspondence identify the straight lines of a ruled surface.

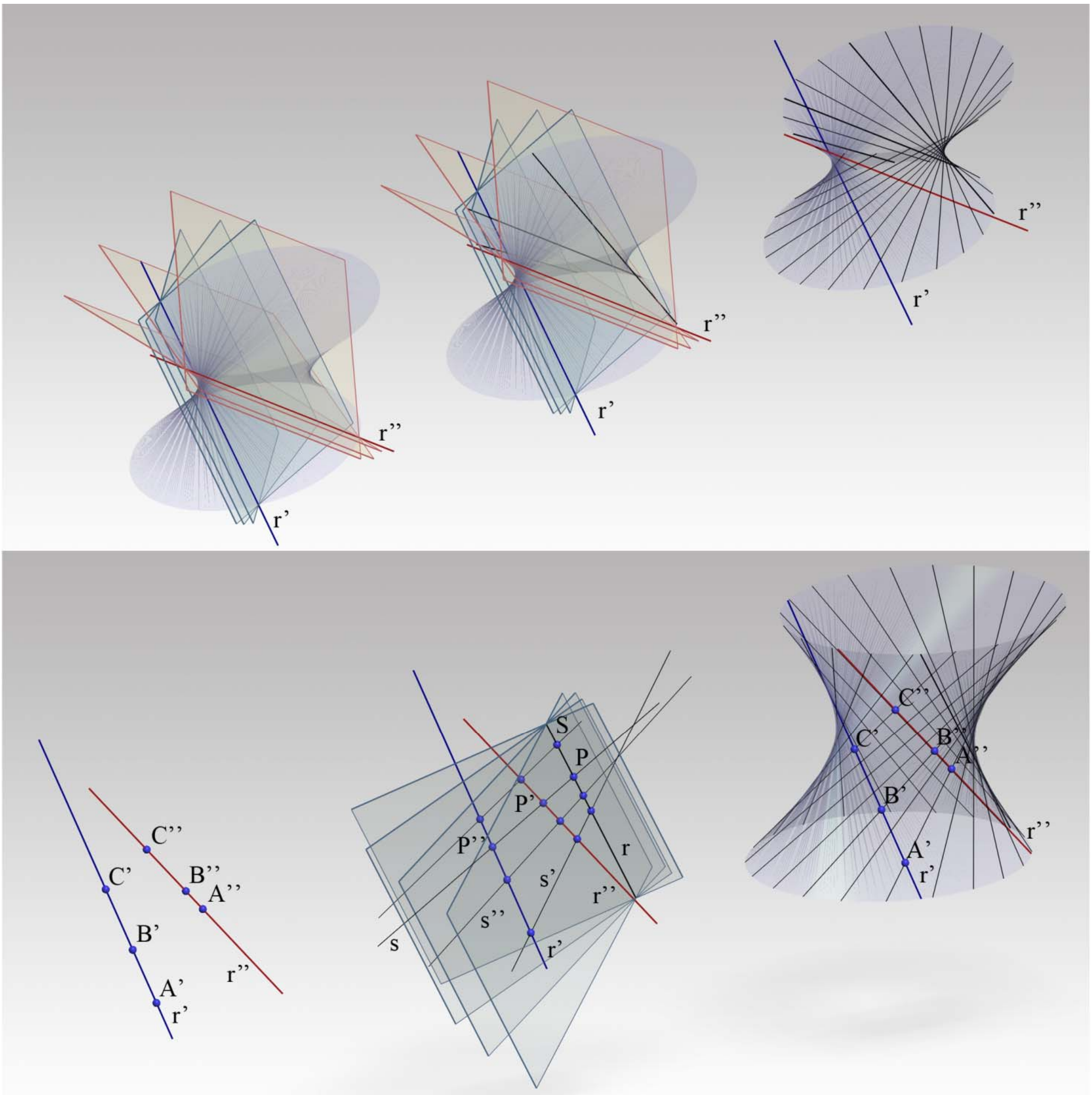

Fig. 5 - Two projective sheaf of planes that generate a ruled hyperboloid. Two projective pencils that generate equally the same ruled hyperboloid

These two statements are dual, as dual are the forms of the first kind, in fact the sheaf of plane is generated by projection, by an axis, of the points of a pencil and, vice versa, the pencil is generated cutting with a straight line the sheaf of plane . Therefore, two ruled surfaces, constructed as above, are equal: the intersection lines of pairs of corresponding sheaf of planes pass through the corresponding points of the two pencils (Fig. 5). The lines are generatrices of the ruled surface, the axes of the two sheaf are the directrices of the ruled. The mathematical method of representation allows to experimentally verify this important property.

Recall that, given three pairs of points in the plane **A'**, **B'**, **C'** and **A'**, **B'**, **C'** belonging to two distinct pencils, you can build the projectivity that makes them correspond in a conic [3]. Similarly, given two skew lines segment range in space **r'** and **r"** and, on them, three pairs of corresponding points **A'**, **B'**, **C'** and **A"**, **B"**, **C"**, we can build the projectivity that intercedes between the two pencils as follows (Fig. 5):

- we construct the line joining two corresponding points, for example, **C'** and **C"**;
- on this line we chose any point **S**;
- We build the lines **A'A"** and **B'B"**, which are skew, because if the two lines were coplanar, the two pencils would be such;
- We build the line r passing through **S** and relies on the two skew lines **A'A"** and **B'B"**, this line is given by the intersection of the planes **SA'A"** and **SB'B"**;
- Finally, we construct the scheaf of plane which has as its axis the line r: each plane of this sheaf cuts the two pencils **r'** and **r** "according to pairs of corresponding points. In particular, the projective correspondence generated by the sheafs that has the line **r** as axis is a perspectivity. Now, it is obvious that the above construction is also the construction of the rifled which has as directrices the three lines: **r'**, **r"** and **r**. In fact, for each plane of the sheaf which has the straight line **r** as axis, are identified, on **r'** and **r"**, two points **P'** and **P"** that belong to a generatrix of the ruled, which, in turn, cuts the line **r** at a point **P**. Therefore, even the line **A'A"** meets the axis **r** at a point **A**, as well as the straight line **B'B"** meets the axis **r** at a point **B**.

The surface generated in the above manner is an elliptical hyperboloid (or hyperboloid in a flap). Note that, if the axes of the two sheaf of plane that generate the ruled are incident, the generated surface is a quadric cone. Also, if the generatrices lines are parallel to a director plane the generated surface is a hyperbolic paraboloid.

We know that the curve projection of a circle is a conic and that two pencils of lines intersect in a conic. Then a conic is a product of two projective pencils. The method of mathematical representation allows to verify experimentally that, given two pencils of prospective lines, it is possible to determine a conic by translating and rotating the two projective pencils in the plane. The intersection of the corresponding lines give rise to corresponding points of the conic.

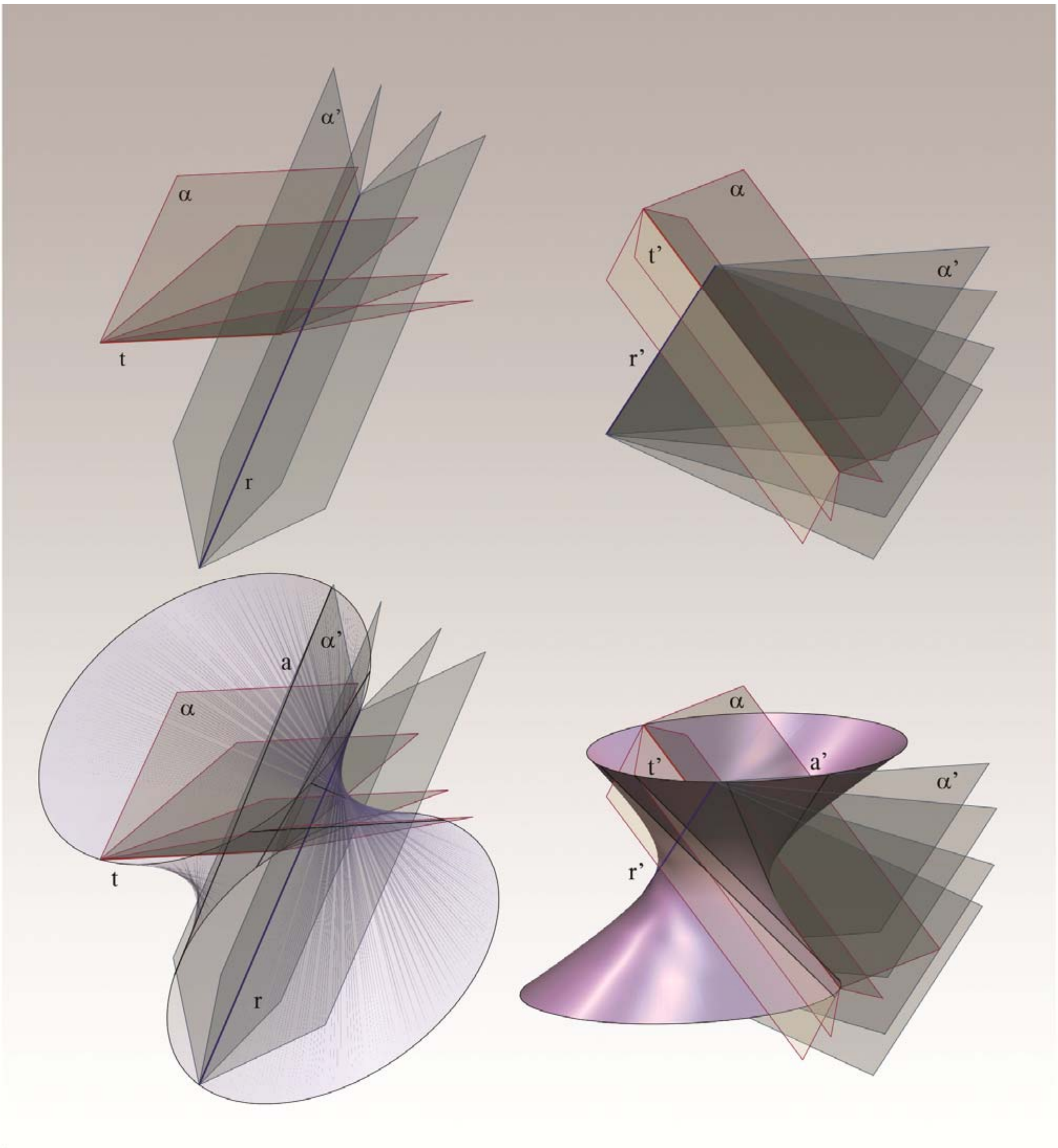

Fig. 6 - Given two sheaf of projective planes, we can freely rotate and translate them into space and always get a quadric ruled as a product of the two new sheaf

The same property belongs to two forms of the first kind perspectives in space. Therefore, given two projective sheaf of planes, i.e. referred to the same range segment line, these give rise to a quadric ruled. And is possible to translate and rotate freely in space two sheaf of planes and get yet another quadric ruled. Take any three skew lines **r**, **s**, **t**. Take the line **s** as the reference pencil and assume the projective sheaf of axis **r** and **t**. Draw at least four pairs of corresponding planes that pass through the points **A**, **B**, **C**, **D** of the pencil **s**. A quadric ruled is determined by three pairs of corresponding planes, so we should take at least another couple of planes in order to experimentally verify the theorem. The intersection of the pairs of corresponding planes gives rise to four straight lines that are four generatrices of the ruled hyperboloid, the two lines r and t are the two directrices. Now we translate and rotate the two sheaf freely in space. Knowing that two projective forms of the first kind remain so if we apply a rigid transformation, we verify that indeed this property is preserved. We rename the axes of the two sheaf in their new positions in space with the letters **r'** and **t'** (Fig. 6). We find the corresponding lines of the two sheaf in their new positions in space. To do this, just find the intersections of corresponding planes. For example, the generatrix **a'** will be the intersection of the two planes α and α**'** corresponding in their new positions, and so on for the other generatrices. We can verify that the two sheaf r 'and t' identify a new hyperboloid scratched. Then, given two sheaf of projective planes, we can freely rotate and translate them into space and always get a quadric ruled as a product of the two new sheaf.

CONCLUSIONS

With the advent of the information technology and the ensuing possibility of constructing virtual spaces of three dimensions, the technology offers today to the researcher and the teacher the possibility to extend the projective constructions to the space. In this paper, we first described the solid perspective in its theoretical basis and in its workability. In particular, we determine the bi-univocal perspective relationship between two spaces: the real space, isotropic, and the contracted and anisotropic space of the solid perspective. We then presented a sphere, studying its projective transformations into ellipsoid, paraboloid and hyperboloid. These transformations were finally examined from the canonical point of view of the projective geometry. Nowadays, thanks to the digital representation, it is possible to experiment directly in space the projective genesis of the ruled quadrics. Given two sheaves of planes corresponding in a perspectivity in space, these determine a surface which is a ruled hyperboloid or a quadric cone. It is possible to untie the two sheaves and freely move them about in space observing that the projectivity is preserved and that the two projective sheaves, in their new positions, determine a new ruled quadric.

The new digital tools therefore offer new opportunities for research and teaching for the synthetic study of the problems of geometry and enhance the heuristic power of drawing.

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