ABSTRACT: The study illustrates the solution of the three point space resection problem, treated by Gaspard Monge in Section V of *Leçons de Géometrie Descriptive*. The problem entails the construction of the intersection curves of three tori. To solve this problem, Monge introduces several simplifications but, nevertheless, makes a mistake; this mistake has already been pointed out by Gino Loria regarding the number of solutions allowed by the problem [11]. The mathematical representation, thanks to its high level of accuracy, today permits not only an efficacious solution of the general case, it also highlights without difficulty the right number of solutions.

We applied this theory to a case of photogrammetric rendering, difficult to carry out by means of the tools offered by commercial software. Case in question concerns the reconstruction of the architectural volumes, now lost, which were located along the road that crosses a village, near Rome. As is known, the reconstruction of points in space from two images is possible if these images are projective and we have at least two projective orientated stars. The first image is a vintage photograph (1892), the second image is a surveyed plan of the masonry still present at the site. Therefore, one of the two projective stars is assimilated to a class of vertical straight lines. With regard to photography, the problem is articulated in two typical phases of photogrammetric processes: internal orientation and absolute orientation. For the absolute orientation we used the pyramid vertex method, in use since the Eighteenth Century, which consists in determining the projection center from three given points of which the positions in space are known.

The solution to the problem posed by the case study can be considered as a useful result. More interesting, however, is the result of the intersection of the three tori with the incident axes (fig. 1). It is, in fact, a graphic process that Gaspard Monge had already proposed in 1798 as a suitable alternative to a system of equations that he considered difficult to solve. In particular, Monge explains how the descriptive geometrical procedure, involving the vision of the represented forms, allows you to exclude in a simple and direct manner the solutions that resolve the problem from theoretical point of view, but do not solve it in the real case because they lead to unrealistic placements of the projection center. Thus, the symbiosis between calculation and analog description, Monge had predicted in these words: «[...] la géométrie descriptive porterait dans les opérations analytiques le plus compliquées l'évidence qui est son caractère, et, à son tour, l'analyse porterait dans la géométrie la généralité qui lui est propre [...]» [18].

Keywords: Descriptive geometry, three point resection problem, Gaspard Monge, photogrammetry.

1. INTRODUCTION

Much has been said and written about the role of Gaspard Monge in the history of representation, sometimes attributing him the 'invention' of descriptive geometry, sometimes reducing, perhaps unfairly, his merit to that of editor of a book already written in the course of previous centuries. But, beyond this difficult evaluation, there is no question that Monge must be recognized as given scientific dignity to the representation of objects in three dimen-
sions. This possibility, distinctly expressed from Frézier as a requirement of engineering and architecture, was not, before Monge, expressed equally explicitly, even in the work of François Sylvestre Lacroix, that also preceded, of course, the publication of the Monge lectures. In his effort to theorize and give general validity to the proceedings hitherto confined to stereotomy and gnomonics (i.e., in special cases), Monge sought a synthesis between the analytical study of geometric shapes and their properties and the graphic and visual representation, therefore, of the same forms and properties, in the belief that the combination of the two methods could give new impetus to the geometry, « [...] Il seroit à desirer que ces deux sciences fussent cultivées ensemble [...] » [18].

Curiously, two centuries after the golden age of Géométrie Descriptive, Monge’s hope seems to be realized in the mathematical representation of the digital universe. In fact, as we have noted several times, the accuracy that the calculation gives digital representation, the possibility of building forms in three dimensions, the possibility of using curved and double-curved surfaces as tools in geometrical construction, combined with the power of visual analogy, allow us to experiment the Monge’s far-sighted vision [13].

This paper traces, using the synthetic method, Monge’s solution, to the problem of the pyramid vertex construction, given its triangular base data and the corners at the top, and showing an original application to the reconstruction of volume reconstruction of a partially lost agglomerate of historical interest.

2. MONGE AND THE THREE POINT SPACE RESECTION PROBLEM

The IV section of the Géométrie Descriptive of Gaspard Monge, entitled Application de la méthode de construire les intersection des surfaces courbes à la solution de diverses questions, is devoted to the solution of six questions, which concern the determination of the position of a point starting from that of other geometric entities [16].

Three of these questions are of theoretical nature and three instead derive from practical problems. In particular, the fifth and sixth questions concerning the construction of the position of a point on a slope, notes that it is the relative positions of the other three points, which are visible from the point sought [11]. In the first case, the tools available are a goniometer and a plumb line, in the second case it is the goniometer only; therefore, we consider the second case, which generalizes the problem.

The point sought, where we imagine the observer was placed, corresponds to the pyramid apex that has its basis in a triangle whose vertices are the three known points on the ground. From his position, the observer can measure, the angles that the projecting lines passing from the three given points, form between them. The problem then consists in determining the position of the pyramid vertex given its base triangle ABC and the angles at the vertex α, β e γ.

We construct the three corners of the triangle AB, BC, AC; let us consider one of this, for example, AB, and then we consider the vertex angle subtended by AB.

We construct a circle on the ABC plane capable of the angle α said. This circle is divided into two arcs from the AB chord. The π−α angle of the AFB arc is supplementary to the angle of the ADB arc. Then, rotate the two arcs of circle around the AB axis and thus generate the double-nappe surface of a torus; we call inner nappe, one generated by the AFB arc, and outer nappe, the one generated by the ABD arc. All points belonging to the two nappes of this torus are seen by the AB segment, under the same angle, α for the outer nappe and π−α for the inner nappe (fig. 1).

We reiterate the procedure for the AC and BC edges of the pyramid base, and we get the other two tori of double nappes, the locus of points seen by the data segments in the respective β and γ corners.

The pyramid apex, by construction, must belong to the surfaces of the three tori, in particular their outer nappes, with respect to the
example proposed in the figure. These nappes intersect each other in three skew curves (one or two branches). The intersections of these curves give rise to a maximum of eight points, eight vertices of a pyramid which can satisfy the conditions imposed by the problem. These

Figure 1: The Monge three point space resection problem, digital synthetic solution.
solutions are all valid from a theoretical point of view, but only one satisfies the problem of a practical nature that we set. Since the eight solutions are symmetrical with respect to the ABC plane, we can exclude the half, in particular those which are found beyond the ABC plane of symmetry. Among the remaining four, one meets the conditions of the observer's position relative to the plane ABC. The synthetic method allows you to directly select the right one.

This question, which has had a significant impact in solving the problems of orientation in photogrammetric surveys, stimulated, from its first publication, the interest of several mathematicians. Among these, Lacroix and Hachette, who dedicated various articles to this problem, published in his major works of descriptive geometry [8-9-10], but also Nichola Fergola and students of his school [10], including in particular Vincenzo Flauti [6-7]. There were many ambiguities on the number of possible and admissible intersections for the solution of this problem, presented for the first time by Lagrange in the Acts of Berlin of 1773. Monge himself, while recognizing eight feasible solutions, sixty-four inferred from the calculation, arrived at a general solution that first Flauti, then Loria reported as incorrect [11-6]. Hachette a few years later re-examined the issue and identified 16 feasible solutions [8], then reduced to 12, in a subsequent article [10] (fig. 2). Hachette, however, never specifies the maximum number of admissible solutions. In fact, the Hachette's solutions refer to the configurations of the problem he represented, and include vertices which contextually observe the given angle and the supplementary angle and therefore, must be discarded. Vincenzo Flauti, in Geometria di sito sul piano e nello spazio, generalized the question by explaining that twenty solutions are found, derive from the combinations, three by three, of the six nappes of the three tori [6]. Of these twenty, twelve do not solve the problem, because they are derived from the intersection of the outer nappes with inner nappes of the same torus and therefore
must be discarded. Therefore eight intersections resolve the problem.

In all the solutions described it is not adequately clear which method identifies the intersections that solve the problem in its generality. It should be noted that this problem is placed at the limit of the possibilities of the synthetic method, when using graphical 2d methods, such as representations of orthogonal projection performed with great skill first by Monge, then by Hachette. Today, the digital synthetic method allows us to immediately track the number of intersections, while in the past, the same 2d graphical method may have given rise to some ambiguities, such as those reported in the Hachette’s text referring to his figures.

3. AN INTERESTING CASE

The procedure described above by Monge finds an interesting application in a particular case illustrated below. This is a reconstruction of the volumes of a paper mill in the village of Grottaferrata, Italy, today lost as a result of an earthquake, from some survey data: a vintage photo from 1892, a picture of AOS 1938 and the relief of the walls still present on the site.

In general, the reconstruction of points in space, from their two-dimensional images, is possible provided that the images are of projective nature and provided that you have at least two projective-oriented stars. In the present situation, the first image is made from a vintage photo (fig. 3), while the second can be reconstructed from an aerial photo of 1938, and by the relief of the remaining walls. Regarding the survey, you can assimilate the star to a class of vertical straight lines, and the same can be done, with good approximation, to the aerial photo, considering: the portion of the flight of AM on the one hand, and the fact that we are considering only a small portion of the frame, on the other. Thanks to the survey, this star may also be consider oriented. Regarding the first image, however, i.e. the photograph of 1892, the problem is more complex and must be broken down into two phases typical of photogrammetric procedures: the internal orientation (position of the center of projection with respect to the frame) and the absolute orientation (frame position in space at the time of shooting). The solution to these two problems permits us the full reconstruction of the projective star and therefore the reconstruction of the location of points in space that no longer exist in reality.

In the photogrammetric restitution, as in any other activity of a scientific nature, you can follow proceedings of deterministic character or probabilistic character. In the first case, having available reliable data and sufficient conditions we arrive at the result without uncertainty. In the second case, when the data are not entirely reliable and conditions are insufficient, you can still achieve a convincing result, through an iterative process, in which the hypothesis that integrates the give data, are introduced throughout the process and the results they provide are compared with the results of previous iterations. In this process we discard the hypothesis that generate, in the iteration, the worst results and keep those that provide the best results, thus converging towards the solution. In this case it is not possible to apply a deterministic process, because the data are insufficient. It is possible, however, to reconstruct the projective star and its orientation in space by successive approximations, converging towards a result corroborated by the reduction of the error. It is, possible, to formulate a

Figure 3: Vintage village photo and 3:2 format.
hypothesis, and comparing it with the results process. The presence of the building volumes still intact in fact, permit us to orient the star and check that the largest possible number of points present in the photograph coincide with the corresponding real points. The hypotheses that will be chosen with respect to others, are those which, in testing, give the best results and the fewest errors. Remember that the interior orientation is known when you know:

- the principal point, i.e. the point at which the camera optical axis intersects the plane of the photographic plate;
- the focal length, i.e. the distance of the projection center (entrance pupil of lens) from the plate.

If the photograph was printed in its entirety, that is, without the cuts that have altered the shape of the plate, and if you have not applied optics and mutual translations or rotations of the plate (as can happen in some professional cameras) the principal point is in the intersection of the diagonals of the frame. In this case, the original format was altered in the press, as can be seen considering the upper left corner of the image, the edge of which is broken. It can therefore be assumed that the section of the higher edge belongs to the original plate, while the lower portion is the result of a cut that was done to approximately align the horizon and the vertical to the edges of the print.

Regarding the relationship between the sides of the original plate it is not easy to formulate a hypothesis because the cameras of the time, and the craftsmanship, used various formats. The archives of Count Primoli (1851 - 1927), for example, are made up mostly of square plates, while many cameras of the time used

![Figure 4: Reconstruction of the horizon line and the principal distance direction.](image-url)
plates whose sides are in the ratio 3:2. The picture we are looking at has sides that are in the ratio 23.9:16.8, which is different from both the 4:3 ratio (i.e. 24:18), both from the 3:2 ratio (i.e. 24:16). Therefore, both hypotheses were verified. We illustrate here the one that has led to better results.

A rectangle was built whose sides are in the ratio 3:2. The left side coincides with the short segment of the original plate which is seen in the upper left. The other sides of the rectangle are such as to include the entire image. We can therefore formulate the hypothesis that the principal point is in the intersection of the diagonals of the rectangle constructed as above. If so, the center of projection will be on the perpendicular to the plate plane and will pass through the principal point (fig. 3).

The cameras of the time used for focus, a sled that allowed a wide range of the lens in front of the plate. Therefore, the focal length cannot be established a priori, even in an approximate way. You can, however, use the knowledge of some corners. In fact, the projection center must be such as to see the direction of the straight line and the arrangement of the buildings plans, from the angle they form in reality.

The most important of these angles is what the plumb direction line makes with the horizontal position. In photography, the direction of the vertical projects into the vanishing point of

Figure 5: The frame internal orientation.
the vertical edges of buildings and can be found with good approximation in point $V'$. Reconstruction of the horizon is not so sure, because the plate is angled, and therefore the horizon does not pass through the principal point, and because it is also hard to find pairs of straight horizontal lines and vertical lines. You can, however, be sure that you will find the horizon above the buildings, and below the limit which separates the woods behind the sky (fig. 4). The checks carried out on various assumptions included in this part of the perspective plane lead to the adopted solution.

You may also consider the angle formed by the two buildings of greatest importance: which include the church (A) and what comprises the turret (B). This angle, deduced from the survey, is equal to thirty-four sexadecimal degrees (or its supplement). Therefore, we construct on the horizon the images of the directions of horizontal lines that belong to the two planes. Then we can construct the points in space capable of these angles, which is a surface of revolution (torus). The projection center is the point where this surface meets the principal distance (fig. 5). This process is repeated until you find the position of the center that satisfies both conditions. The focal length is determined in relation to the size of the frame. In the hypothesis that the original plate measure 6x9, the focal length is equal to 151 mm, length that is compatible with the optical construction of the time, being equivalent to that of a 60 mm lens on the Leica format.

The pyramid vertex method is used for the absolute orientation. The position of the projection center coincides with the apex of a pyramid that sees three points still present on the site, location unknown, under known angles. The instrument used for the angular measurement is, in this case, a vintage photograph that, once oriented, is capable of restoring the angles formed by the straight lines that coincide with the points photographed. The control points, instead, were chosen on the basis of survey data, rather distant from each other, to minimize error. In particular, $E'$, $I'$ and $L'$ were chosen, seen from the projection center under the following angles expressed in sexadecimal degrees (fig. 6):

\[
\begin{align*}
E'O'I' &= 17.7155 \\
I'O'L' &= 23.7762 \\
E'O'L' &= 8.7568
\end{align*}
\]

The solutions of the problem are found through the intersection of geometrical loci of the points that see the data three point below the measured angles (fig. 7). In this case there are, in all, four options, three of which have a purely theoretical significance because they are the vertices placed below ground and the opposite vertex of observer site. The center of the projective star generated by the photographic shooting lies instead in the vertex that represents a possible and coherent position with the image (fig. 8). The procedure for absolute orientation is completed by placing the system formed by the plate and the projection center in the center as constructed above and imposing the collinearity of the edges of the pyramid with the three straight lines connecting the projection center $O$ to the points $E'$, $I'$ and $L'$, which are the photographic image of the real ones $E$, $I$ and $L$ (fig. 9).
Note that performing this operation, the system also requires a precise orientation in space and therefore the straight line projecting from the center in the vertical direction must be vertical, or at least very close to verticality. In the present case the straight line form with the vertical an angle of 1.6 degrees sexadecimal. The residual errors can be calculated collimating these points. In this case we obtained errors of decimeter magnitude.

4. CONCLUSIONS

The solution to the problem posed by the case study can be considered a useful result. More interesting, however, is the result of the intersection of the three tori with the incident axes. It is, in fact, a graphic process that Gaspard Monge had already proposed in 1798 as a suitable alternative to a system of equations that he considered difficult to solve. In particular, Monge explains how the descriptive geometrical procedure, involving the vision of the represented forms, allows one to exclude, in a simple and direct manner, the solutions that might resolve the problem from a theoretical point of view, but do not actually resolve it, because they lead to unrealistic placements of the projection center. Thus, the the symbiosis between calculation and analog description, Monge had predicted in these words: «[...] la géométrie descriptive porteroit dans les opérations analytiques le plus compliquées l'évidence qui est son caractère, et, à son tour, l'analyse porteroit dans la géométrie la généralité qui lui est propre [...]» [18].
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